From Static to Dynamic Mathematics: Historical and Representational Perspectives

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1 Preface

We present new theoretical perspectives on the design and use of digital technologies, especially dynamic mathematics software and "classroom networks." We do so by taking a more contemporary perspective of what can be possible, as the notations, the mathematical experiences, and the medium with which these all work, come closer together and co-evolve. In effect, this approach takes a more applied epistemological stance to the nature of mathematics education in the future versus an epistemological tension between the contemporary mathematician and "their" mathematics, and society today.

Kaput began to take a deep appreciation of the evolution of sign systems in the mid-1990s—producing diverse perspectives on the semiotics of mathematical notations for education (Kaput, 1999; Shaffer and Kaput, 1999) and later in Moreno-Armella and Kaput (2005) and Kaput et al. (2008). In these works, he incorporated the evolutionary and cultural perspectives of Merlin Donald's two seminal works (Donald, 1991; 2001) focusing on cognition and representations. Shaffer and Kaput (1999) suggest that the "new phase" (virtual culture) is a logical next-step in the development of the evolutionary-cognitive perspective developed by Merlin Donald. This perspective considers that the evolutionary study of cognition can be conceived of as a timeline going from the mimetic culture, then mythic culture (orality) to finally, the theoretical culture, based on external memory supports. This includes

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writing as its main component. They suggest that virtual culture, based on the processing capacity of new technologies (and future ones), is the next logical step. As their perspective is mainly cognitive-evolutionary, they do not study in depth the semiotic aspects of this new stage in the evolution of cognition. We approach the study of notation systems from a historical perspective emphasizing the semiotic dimension. This leads us to a serious consideration of the dynamic structure of the reference field. This is in contrast to Shaffer and Kaput (1999), which emphasizes the processing power of the new notation system due to the computational embodiment. We build now on this work in considering the epistemological transformations due to the presence of the executability of digital semiotic representations. This has allowed us to cast light on historical stages of evolution and compare them with present stages of technological evolution in mathematics education (Kaput, 2000). In this chapter, we propose a new way of conceiving a reference field in mathematical activity. This reference field becomes possible through certain new technologies and hardware infrastructures, particularly wireless networks.

2 Perspectives on Symbols

A symbol is something that takes the place of another thing. For instance, *pencil* takes the place of those material objects we use to write. But this does not still explain how *taking the place* occurs in general. A symbol is something that someone *intends* to stand for or represent something other than itself. Symbols *crystallize* intentional actions, and are instrumental for generating and developing human cultures. We wish to use the metaphor *crystallize* instead of encapsulate, because we want to direct our attention to essential properties of a crystal, particularly its stability and its possibility for change and growth.

Generalization and symbolization are at the heart of mathematical reasoning in the SimCalc learning environment. One way a person can make a single statement that applies to multiple instances—a generalization—without making a repetitive statement about each instance is to refer to multiple instances through some sort of unifying expression that refers to *all* of them at once, in some unitary way—as a single thing. The SimCalc program of development has attempted to establish such through linked and multiple-representational software environments with integrated curriculum that attends to and exploits such affordances in order to focus the attention of learners in mathematically meaningful ways.

The unifying expression requires some kind of symbolic structure, some way to unify the multiplicity, and this is the focus of our attention to such a problem in mathematics education. Symbolization is in the service of generalization—both within individuals and historically as communal thinkers. And once a level of symbolization is achieved, it becomes a new platform on which to express and reason with generally, including further symbolization (see Kaput et al., 2008, p. 20).

Now, the difference between different modes of reference can be understood in terms of levels of interpretation (Deacon, 1997, p. 73). We should emphasize that the nature of reference is relative at its core, and what is a referent in one description may be the result of a prior symbolization. These levels of interpretation call attention to *icons*, *indexes*, and *symbols* as used by C. S. Peirce (Deacon, 1997, p. 70). Hence, where we take up the description has a lot to do with "what is a symbol" and "what is a referent" for that symbol.

The idea of crystallization does not imply a rigid and/or static structure for the reference field. Instead, with this perspective, reference fields (meanings) are *dynamic*—they grow and transform with the shared use of symbols. The reference field lodged in a symbol can be greatly enhanced when that symbol is part of a network of symbols.

Emergent meanings come to light because of the new links among symbols. For instance, the meaning of a word, in a dictionary, can be found inside the net of relations established with other words. Nevertheless, Donald (2001, p. 154), has suggested that as our early experience is gained in a non-symbolic manner, the roots of meaning can be found in our non-symbolic engine, that is, in our analogue modes of operation, as if the ultimate meaning of a symbol were an experience, an intuition. Yet, we have been able to create symbolic universes that *duplicate* our experience and provide a meta-cognitive mirror where we can see ourselves and enrich our lives and thinking. This is the case, for instance, with works of arts, novels, and scientific theories. The feeling of objectivity that comes with our symbolic creations explains the Platonic viewpoints of many scientists. In mathematics, this viewpoint translates into the belief in a pre-symbolic mathematical reality. Explaining the mathematical power embodied in Maxwell's equations for electromagnetism Hertz wrote:

> One cannot escape the feeling that these mathematical formulas have an independent existence and intelligence of their own (Kline, 1980, p. 338).

This crystallizing impact of symbols in our minds generates the belief that they are the primordial world of experiences in the first place. But if we are doing mathematics, for instance, we need *some* conviction that we are working with objects that have a real existence even if this existence is not material existence. This has been a central philosophical and theoretical focus of the mathematical tapestry of the SimCalc learning environment.

Platonism becomes acceptable only as *emotional* Platonism. However, in absence of symbolic representations, we lose the access to mathematical objects, as they are intrinsically symbolic objects. We can speak of mathematical inscriptions as the external marks of symbols but we cannot forget that symbol and reference are like a one-sided coin—each one is the condition of existence of the other. Before, we said that a symbol crystallizes an action or an intentional act. What kind of action is crystallized in a mathematical symbol? We consider this question central for the epistemology of mathematics and mathematics education, which the SimCalc project has problematized over several decades.

Later in this chapter, we analyze how new forms of mathematical activity—through dynamic media—appreciates this fundamental perspective for allowing students more direct access to mathematical structures. We will present examples of classroom activities to illustrate the dynamics of this process, but first we aim to cast light on the evolving nature of the relationships between a mathematical symbol and its reference field using some historical examples.

Incised bones like the one found in Moravia (Flegg, 1983), dated 30,000 B.C., constitute what is perhaps the first example of manmade symbols. We interpret this finding as an example of the use of a one-to-one correspondence between a concrete collection of objects (perhaps preys attributed to a hunter) and the set of incisions on the bone. This set of incisions, acquire a symbolic meaning. In fact, the act of incising a bone is an *intentional* act by means of which the bone is modified to store, manipulate, and transport information—-an incision, for instance, can represent a rabbit, a bird. On the bone, one can see after this intentional act the birth of a new symbolic world—-the *territory* of the symbol. Tokens are our next example in the production of mathematical symbols. As D. Schmandt-Besserat has written in her fascinating account on *How Writing Came About* (1996) tokens were "small clay counters of many shapes which served for counting and accounting for goods" (p. 1). Tokens served the needs of economy and their development was tied to the rise of social structures (Schmandt-Besserat, 1996, p. 7). After a few decades as trade increased, Sumerians needed a more compact way to keep track of goods than individual tokens. Thus, the tokens which according to shape, size, and number represented different amounts and sorts of commodities—were put into a sealed envelope, a container for the tokens. This process compacted information but created a new problem: to inspect the content of an envelope, it had to be destroyed. This new problem was resolved, as Schmandt-Besserat recounts (1996, p. 7) by *imprinting* the shapes of the tokens on the surface of the envelope. A mark impressed on the surface of the envelope kept an *indexical* relation with one counter inside, which figured as its referent. After another one hundred years, Sumerians realized that they could dispense with the tokens themselves by just impressing them on wet clay. In fact, transferring their conventional meaning of the tokens to those external inscriptions was enough to convey the information intended (Schmandt-Besserat, 1996, pp. 50–51). That decision altered the semiotic status of those external inscriptions. Afterwards, scribes began *to draw* on the clay the shapes of former counters. But drawing a shape versus impressing the shape of a token are extremely different activities, even if both are intentional. This gradual, emerging set of physical inscriptions worked as a meta-cognitive mirror to guide actions—both mental (interpretive actions) and physical (elaborations)—on the new inscriptions.

As Duval (2006) has explained,

> One has only to look at the history of the development of mathematics, to see that the development of semiotic representations was an essential condition for the development of mathematical thought. For a start, there is the fact that the possibility of treatment, for example calculation, depends on the representation system (p. 106).

Duval (2006, p. 107), explains as well that the crucial problem of mathematical comprehension for learners arises from the fact that the access to a mathematical object is possible only by means of semiotic representations and yet that these representations cannot be confused with the object itself. In fact, each time we produce a new system of representation for a mathematical object, that object is no longer the same object. Mathematical objects have many potential faces and each face corresponds to a certain way of operating the object. Mathematical objects are always under construction. This construction takes place within symbolic cultures, as happens with novels and sonatas, for instance. The importance of notation systems (semiotic representations) cannot be overemphasized. Reading classical mathematical texts from the remote past, one can appreciate how after translating those texts into modern notation, the problems become almost trivial. This is the case, for instance, with arithmetic problems from pre-Greek mathematical cultures. Were these trivial problems? No. Reflecting on these issues, one arrives at the conclusion that mathematical notation systems are not epistemologically neutral. It must be taken into consideration that notation (or semiotic) systems are artifacts coextensive with our thinking. We say we think with a notation system when we use it as a cultural tool. For instance, when we compute using the binary system for numbers we feel that system is outside of our mind. But if we compute with the decimal system, the feeling is quite different. It is as if this system were an intrinsic component of our mind. And it is, in fact, because a process of internalization has taken place. The system has gone from the (school) culture into our mind. It becomes coextensive with our mind. We think through it. Vygotsky considered the process of internalization—cultural artifacts becoming cognitive tools—central to his theory of cognition. He said "any higher mental function is external because it was social at some point before becoming an internal truly mental function" (Wertsch, 1985, p. 62).

3 Shifting from Static to Dynamic Media for Twenty-First Century Classrooms

The visual, gestural, and expressive capacity of the use of new technologies becomes apparent in various ways. These capacities primarily focus on the medium within which the technology user, learner or teacher (from hereon described as the user) operates. To describe this change, we introduce the idea of co-action to mean, in the first place, that a user can guide and/or simultaneously be guided by a dynamic software environment. This is basic in understanding that humans-with-media (see Borba and Villarreal, 2005) is a fundamental development in the co-evolution of technology and educational environments.

Notation systems have evolved in new ways with new mathematical explorations, hitherto impossible or impractical in the static medium. This stage of mathematical epistemology is presently situated in the education domain, and less so, if at all, in the mathematician's domain.

We suggest that the evolutionary transition from static to dynamic inscriptions, and hence new forms of symbolic thinking, can be modeled through five stages of development, each of which can still be evident in mathematics classrooms in the twenty-first century.

3.1 *Stage 1. Static Inert*

In this state, the inscription is “hardened” or “fused” with the media it is presented upon or within. Even though this historically has been how ancient writing was preserved (e.g., cuneiform art, bone markings) it is also the description of many textbooks and handouts from printers in today’s classroom. Early forms of writing can even include ink on parchment, especially calligraphy as an art form of writing since it was very difficult to change the writing once “fused” with the paper. In this sense, it is inert.

3.2 *Stage 2. Static Kinesthetic/Aesthetic*

With the advance of scribable implements and the co-evolution of reusable media to inscribe upon, we enter a second stage of use, categorized by erasability. Here, chalk and marker pens allow a transparent use of writing and expression, as their permanence is temporal, erased over time. But, this new form, albeit static, affords a more kinesthetic inscription—given it is easy to move within the media of inscription—and an aesthetic process—given the use of color to differentiate between notations.

3.3 *Stage 3. Static Computational*

As the media within which the notation system is processed and presented changes we observe a third stage of evolution. Here presentations (e.g., graph-plotting) are artifacts of a computational response to a human’s action. The intentional acts of a human are computationally refined. A simple example is a calculator where the notation system (e.g., mathematical tokens, graphs, functions) is processed within the media and presented as a static representation of the user’s input or interaction with the device.

3.4 *Stage 4. Discrete Dynamic*

As computational affordances make the medium less static, and user interactions become more fluid, the media within which notations can be expressed becomes more plastic and malleable. The co-action between user and environment can exist. This process of presentation and examination is discrete. For example, a spreadsheet offers an environment within which a user can work to represent a set of data by different intentional acts, e.g., “create a” list, “chart a” graph, “calculate a” regression line, or is generated through parametric inputs, e.g., a spinner or a slider alters some seed value. Both of these discretize actions into observable expressions—

expressions that are co-actions between the user and the environment—yet the media is still dynamic, as it is malleable, and re-animates notations and expressions on discrete inputs.

3.5 Stage 5. Continuous Dynamic

This stage builds on the previous stage by being sensitive to kinesthetic input or co-action, to make sense of physical force, or gestural interaction through space and time. Some software allows the user to navigate through continuous actions of a mouse—the perception or properties of a mathematical shape or surface through re-orientating its perspective, e.g., what does this surface look like when I click/drag and move the object? Haptic devices can detect motion through space and time, and provide feedback force on a user's input. For example, a user could perceive the steepness of a surface through a force-feedback haptic device and move it to a point of extreme value without asking the computer to calculate relative extrema.

4 Dynamical Perspectives of Mathematics Reference Fields: Variation and Geometry

The nature of mathematical symbols has evolved in recent years from static, inert inscriptions to dynamic objects or diagrams that are constructible, manipulable and interactive. Learners are now in a position to constitute mathematical signs and symbols into personally identifiable objects, and systems of objects. The evolution of a mathematical reference field can now be an active process that learners and pedagogues can both assist in, can identify with and can actively update. Hence, the reference field has the potential to co-evolve with human symbolic thinking. We will use examples from new innovations in technology to illustrate how work in dynamic mathematical environments (mainly software but also one example which combines both software and hardware) allow new avenues for learners to be actively involved in the evolution of new reference fields.

4.1 Variation and Geometry (Co-action and ZPDA)

Mathematical objects are crystallized through diverse symbolic representations. At prior stages, we only had inert symbolic systems. Those are found in printed books, for instance, and still continue to be instrumental in mathematics at school and research levels. Crystallization is a process with social and cultural dimension. Today, mathematical objects are undergoing another level of crystallization as they migrate to screens and other media where symbols and representations are executable.

We will use examples from recent innovations in digital technology to illustrate how working in dynamic mathematical environments opens new perspectives for

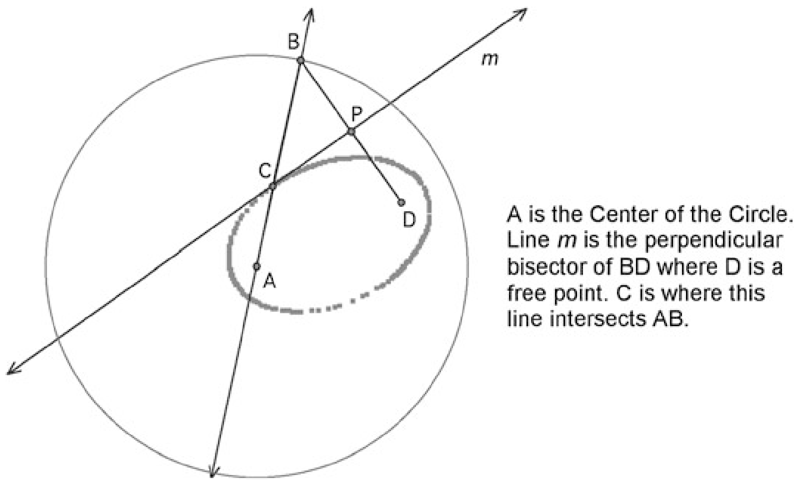


Fig. 1 Locus

learners. Dynamic geometry environments offer point-and-click tools to construct geometrical objects. These can be selected and dragged by mouse movements in which all user-defined mathematical relationships are preserved. In such an environment, students have access to conjecture and generalize by clicking and dragging hotspots on an object, which dynamically re-draws and updates information on the screen as the user drags the mouse. In doing so, the user can efficiently test large iterations of the mathematical construction.

Figure 1 attempts to illustrate this dynamism through a snapshot of such a physical action. As point B is dragged by the user the environment updates by presenting a dynamic representation of all possible iterations of the construction, or "solutions" to the constraints of the construction, i.e., the locus of point C (an ellipse in this case). The nature of the construction constrains the path of C to an ellipse. Since triangles PCB and PDC are congruent, $CB = CD$. $AC + CB =$ constant, so $AC + CD =$ constant. This is the original, static definition of the ellipse. Now, something else comes to the frontline, the enhancement of the mathematical expression, through the animation of point B. As the point C is structural to the construction (it is always updated as the intersection of the bisector m with line AB) it follows the elliptical path.

Indeed we have discretized this "physical" motion. But what we have here is an illustration of where the user has not only actively constructed the ellipse, but has the affordance of a flexible media where the diagram can be deformed with the engineering preserved, through one dynamic action. The dynamic action allows a series of constructions to be instantly created as an embedded environmental automated process. For instance, if we take the point D outside the circle, the ellipse becomes a hyperbola making tangible the intrinsic link between these conic sections. Here the system of tools are embedded and the field of reference (for the symbolic representation of the conic section) is being broadened because the structural points in the construction, due to the possibility to be re-placed in the (digital) plane, lodge new meanings into the executable structure. The dragging of structural, well-constructed objects enables the user to establish whether the mathematical constructs that underlie its engineering can be preserved upon manipulation. Once this is done, the user is enabled to flexibly explore the digital object—-which embodies a mathematical structure. This possibility translates into another dynamic perspective on geometric diagrams and is referred to as a "drag test." Such embodied actions of pointing, clicking, grabbing and dragging allows a semiotic mediation (Falcade et al., 2007; Kozulin, 1990; Mariotti, 2000; Pea, 1993) between the object and the user who is trying to make sense, or induce some particular attribute of the diagram or prove some theorem. Once again, the reference field co-evolves with the user's symbolic thinking and/or reasoning. The kinesthetic actions of the user are crystallized within the geometric diagram and they become part of the enriched, new, mathematical object. The conic is not anymore a definition whose visual trace appears magically on the page. Now, the user *sees* the conic emerging from the screen through her actions now coextensive with the executable system of representation. The user is *co-acting* with the medium, her intentionality is embedded in there and the answer arrives as a digital gesture: the conic on the screen. The plasticity of these actions, the mutual transformation of medium and user, is much more than the classical *inter*action between a user and a rigid tool. The media can keep a trace of such constructions and actions and the user is allowed to rehearse the whole event. The diagram is crystallized in the digital medium but the virtual realities of the diagram obey the rules of geometry that are preserved in the elements of the diagram, just as world objects obey the rules of physics in nature (Laborde, 2004). Again, this sense of reality that the user feels becomes an important element in her cognitive space. We call this certainty *emotional* Platonism. When an element of a diagram is dragged, the resulting re-constructions are developed by the environment NOT the user.

Formalization and rigor are relative to the media in which they take place. They have to respect the nature of these media. If we use digital semiotic representations of mathematical objects, what are the new rules *to prove* a theorem, for instance—that are considered *legal* in the new digital environment? This methodology is highly dependent on time. We can find traces of it in the works of the greatest mathematicians of the past. Euler, for instance, is the author of proofs that could not be published today. Mathematics has been continuously transforming its standards of proof.

As we have seen above, the *executability* of digital semiotic representations of mathematical objects broadens mathematical expressivity. By allowing the externalization of certain cognitive functions (graphing a function or finding the derivative, for instance), executability makes possible the co-action of the student with a digital environment. How is this to affect mathematics education in the future? Kaput et al. (2007, p. 174), observed that the inherited corpus of shared mathematical knowledge produced in interaction with pre-digital technologies is large and stable. So, we need to create early *transition strategies* to transform basic contents of this stable corpus of mathematical knowledge into the new digital semiotic supports. For instance, take the Hilbert space-filling curve—a continuous fractal curve first described by David Hilbert in 1891. In Fig. 2, we illustrate the recursive process that renders the curve as the limit of the sequence.

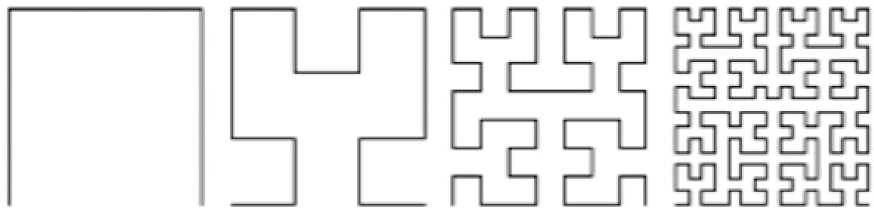


Fig. 2 Hilbert space filling curve

To prove the original theorem, following classical methodology, is an intricate task. However, when one turns the result into a digital one—writing an executable Logo procedure, for instance—we can arrive at the following version: *Given a (screen) resolution, there is a step in the recursive process that generates the curve that fills that screen.*

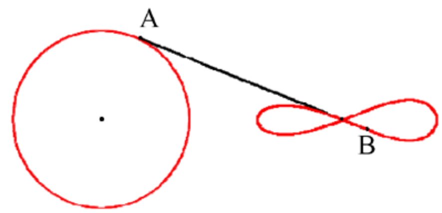
We can imagine the sequence of recursive levels that generate increasingly better approximations to the Hilbert curve, as metaphorical levels of crystallizations of this mathematical object. When the only recourse the students have to penetrate into the mathematical complexity of this object is the analytical representation, we are rather sure that it will remain hidden from their eyes. Now, the executable representation from the digital medium comes as a substantial mediation artifact for students in terms of their cognition. We do not intend to say that this version is formally equivalent to Hilbert's original version of the theorem. In fact, the digital version is a different result. The digital version opens a window into the former version and, at the same time, suggests what is in the future of the students: the meaning of the theorem. René Thom (1973) said it with these words:

> The real problem that confronts mathematical teaching is not that of rigor but the problem of the development of *meaning*... (p. 202).

In the case of Hilbert's theorem, the digital, executable representation is an artifact for developing the meaning hidden in the original analytic representation. Knowing what a mathematical object entails, we need to find and construct the web of relationships among a diversity of previous symbolic instantiations of the mentioned object. In the present example, the executability of the procedure and the role of the digital medium, make the mathematical object tangible. Knowing the resolution, we can calculate the step in the recursive process that will fill the screen. This is an unexpected activity made possible by the new instantiation of the theorem—one which might provide educational meaning. Here we are still working at the border between the paper and pencil (classical) epistemology and the new digital (applied) one. Courant and Robbins—in their classic *What is Mathematics?*—advocate the role of intuition as the driving force of mathematical achievements. And intuition becomes reinvigorated when mediated by digital media as these provide a strong visual component for mathematical thinking. Visual, dynamic perception offers an opportunity to extend mathematical interpretation and, following Thom, *meaning*.

In his *Remarks on the Foundations of Mathematics*, Wittgenstein (1983) emphasizes the role of the eye whilst describing a sketch of a mechanism for drawing

Fig. 3 Wittgenstein figure eight



curves: "when I work the mechanism *its movement proves the proposition* to me; as would a construction on paper" (italics added, p. 434).

He was thinking of this as a mechanism to draw a figure eight as shown in Fig. 3. The digital version of the mechanism makes it ostensible that it is probably more powerful than Wittgenstein originally thought. In fact, by changing the length of segment AB one obtains a beautiful family of curves, full of plasticity, and unfolding continuously on the screen. The unfolding process itself makes explicit the intimate connection among these curves a fact that, in a static medium, results invisible for most students. *Exploring through movement* becomes a new tool for the students. Let us exhibit three stages in the evolution of the figure eight coming from Wittgenstein *digital* machine (see Fig. 4).

The unfolding process takes place as the segment AB is lengthened. The mathematical object under study is not any longer a remote, static object. The immersion in the digital medium provides the students with extended resources to explore and articulate their mathematical reasoning.

We will dedicate the remainder of this chapter to explain and substantiate the thesis that dynamic, digital technologies have the potential to transform the infrastructure of the mathematics classroom, in particular, the distributed cognitive and communicative activities. As Rotman (2000) has forcefully suggested:

> Such a transformation of mathematical practice would have a revolutionary impact on how we conceptualize mathematics, on what we imagine a mathematical object to be, on what we consider ourselves to be doing when we carry out mathematical investigations, and persuade ourselves that certain assertions, certain... a "theorem" for example would undergo a sea change (p. 68–69).

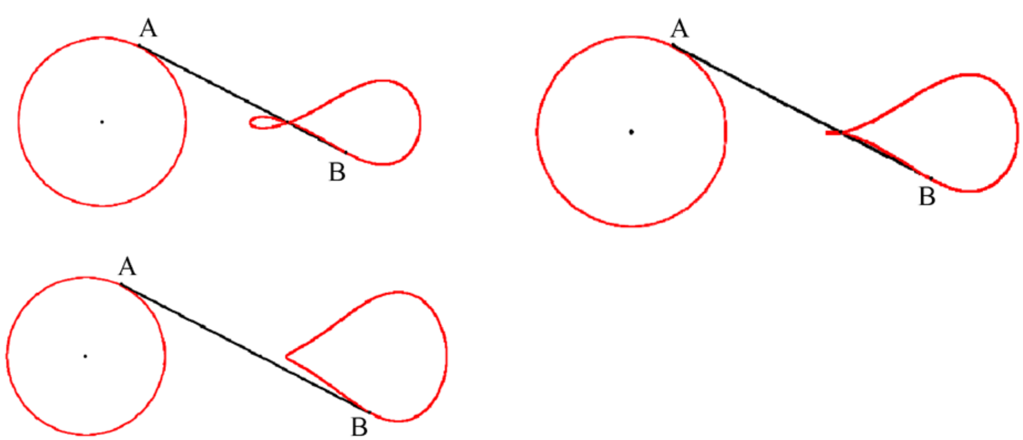


Fig. 4 Stages of evolution of the figure eight

4.2 The Case of Multiple Representations in a Networked Context

Dynamic representations or representations that can be executed either by the user or the environment are available in algebraic contexts as well as geometry. We now continue to illustrate the change in the evolution of reference fields as one shifts from a static to dynamic media and how exposure to such environments can contribute to the fusion of multiple forms of function hitherto loosely connected. SimCalc MathWorlds®—hereon called SimCalc—supports the creation of motions via linear position and velocity graphs, which are visually editable—by clicking on hotspots—as well as algebraically editable. These motions are simulated in the software so that users can see a character move whose motion is driven by the graphs they, or someone else, have constructed. Students can step through the motion and perform other operations in order to help them make inferences about an algebraic expression to represent the graphs. Software runs on hand-held devices (for example, the TI-83/84Plus graphing calculator or the Palm) as well as across computer platforms (as a Java Application). In SimCalc, users can interact with simulations of phenomena. In addition to three traditional core representations of mathematical functions (tables, graphs and formula), motion becomes a conduit to allow fusion of these forms. Motions can be created synthetically within the environment or physically through the use of a motion detector. Consider the following example: Your friend is walking at 2 ft/s for 10 seconds, you need to create a motion that starts where she ends her motion and ends where she starts. Motion data would be represented as a position time graph, but an additional feature is that your motion can be re-played in SimCalc.

Now your contribution to the environment is personally meaningful, and a fusion between traditional "engineered" forms of functional forms, i.e., graphs, and personal mathematical motions occur. Your motion is a form of semiotic embodiment since your motion is mathematical and provides a facilitator for mathematical symbolism. Re-enacting the phenomenon is yet another form of executable representations that allows users' intentions to become crystallized into new, examinable mathematical symbols. Developing understanding of core algebraic ideas such as slope as rate and linear functions ($y = ax + b$) is an important piece of mathematics in which such an environment focuses. Objects in SimCalc are referred to as actors in associated curriculum. Marks indicating where an actor is—at specified intervals of time—can be a feature that one can use in SimCalc.

The actor's motion is preserved or crystallized in this set of marks and the slope of the associated graph is also crystallized in this set of marks. Indeed, it is a new, erasable (through resetting the simulation) inscription that informs the viewer that the actors are moving at constant rates (at least from second to second) and at a speed (or rate) of 2 ft (the gap between marks) per second. The slope of the graph is this rate—2 ft/s—and so it is a representation of a rate graph (velocity) that would be associated with this motion. Links between position (accumulation) and velocity (rate) graphs is a fundamental calculus principle that is being made accessible through executable representations in SimCalc (Nemirovsky and Tierney, 2001; Nemirovsky et al., 1998). Such work by Nemirovsky and his colleagues has shown that students can make sense of dynamic time-based graphs and connect these with certain ideas and skills in arithmetic, e.g., number sentences involving addition and subtraction.

We conclude with a brief, yet synergistic example, based upon recent work on classroom connectivity. Our work (Hegedus and Kaput, 2002, 2003; Roschelle et al., 2000) has combined the power of dynamic mathematics with connectivity. Here mathematical constructions that individual students have created are aggregated into a public workspace via a communication infrastructure (for example, using internet protocols or wireless networks). The computer version can send constructions to other computers and receive constructions from various TI graphing calculators running SimCalc software (see http://kaputcenter.umassd.edu and education.ti.com).

In allowing this, students can create families of mathematical objects that interact in mathematically meaningful ways with a well-structured activity. An example of such an activity is a Staggered Race that requires the students to first attain a count-off number within a group. Such an activity exploits the naturally occurring physical set-up of the classroom by segregating the whole class into numbered groups where students within each group are assigned a count-off number. So students can have a unique identifier both in terms of their group and their place in a group. This number is critical to the establishment of structure in the activity and contextualization of the student's construction within the aggregation of the complete class of functions. In this example, each student starts at three times their count-off number but "ends the race in a tie" with the object controlled by the target function $y = 2x$ (so the target racer moved at 2 ft/s for 6 seconds and started at zero—see the bottom graph in Fig. 5). Students now need to calculate how fast they have to go to end the race in a tie.

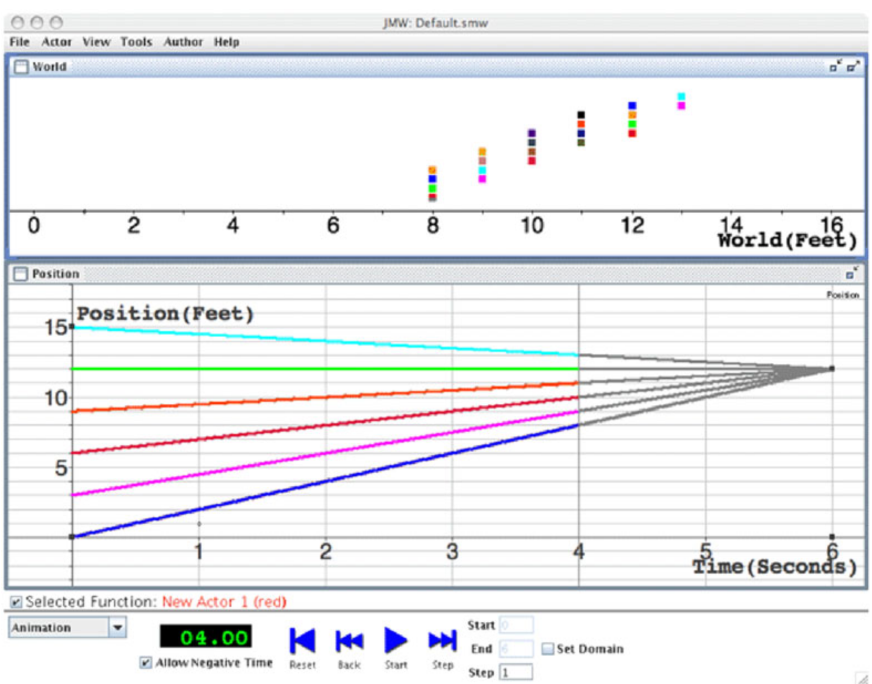


Fig. 5 Staggered start, simultaneous finish

Since they start at different positions, the slope of their graph changes depending on where they start, which in turn depends on their personal count-off number. Secondly, and more importantly, the Count-Off Numbers 4 and 5 give rise to two important slopes (or gradients). The person with Count-Off Number 4 has a graph with constant slope (or gradient), $y = 0x + 12$, since he starts at 12 ft, which is the finish line, so he does not have to move! The person with Count-Off Number 5, starts beyond the finish line (15 ft) and so has to run backwards, thus forcing the student to calculate a negative slope.

In simpler examples, the display of $y = ax$—with a varying as a slope of a family of linear functions—has been described by students as a collection of outwardly spreading motions or a "fan" of functions using a graphical representation. In addition, they use a gestural metaphor via a physical splayed-out finger representation or a set of external artifacts—i.e., a bunch of pens—to display their personal construction of a family of linear functions.

It is through such work that we propose that these digital environments include the social structure of the classroom. Groups of motions and associated graphs can be analyzed both individually and collaboratively. Here the technology serves not primarily as a cognitive interaction medium for individuals, but rather as a much more pervasive medium in which teaching and learning are instantiated in the social space of the classroom (Cobb et al., 1997). Mathematical experiences emerge from the distributed interactions enabled by the mobility and shareability of representations. The student experience of "being mathematical" becomes a joint experience, shared in the social space of the classroom in new ways as student constructions are aggregated in common representations.

Lave and Wenger (1991) have made clear the centrality of legitimate peripheral participation in learning. In the classes that we have been designing and studying, we have deliberately exploited the social situatedness of student learning and likewise the conversational resources for learning (Donald, 2001; Roschelle, 1992, 1996). However, our work has revealed, particularly in the context of currently available communication technologies, that the basic aggregation participation structures as described above have hard edges and little room for legitimate peripheral participation. A student either makes the function for uploading into the aggregate, or not, and the salient presence or absence of the student's contribution is a central rather than a marginal contributing factor to the power of the approach. Hence, we have a design tension requiring creative responses, both in task design and in pedagogy. A simple example of a small change is to have a group assign its own numbers instead of simply counting-off, which offers opportunity to discuss numbers that might be special in the construction (e.g., in the "Simultaneous Finish" situation, to choose 4 as your number so you start at the finish line).

As we discussed earlier with different examples, the reference field now "enlarges" through experience or proximal development of the participant in such an environment, until the symbol and the reference field become the same. Mathematical or theoretical referents are now very individual and personal. The pervasive medium of "connectivity allows the aggregation of dynamical objects into a "dynamic mathematical symbol." Crystallization of individual contributions into a gestalt of dynamic inscriptions occurs with rapid evolution in such an environment. Crystallization embodies the mathematical symbol (in this case, a family of functions) and it is shared across a social space. As the aggregation of individuals' construction are built, shared, and executed, a pathway is laid for mathematical reasoning, abstraction and discovery.

5 Conclusion: New Theoretical Perspectives

Mathematical thinking cannot be achieved exclusively through written symbols; the production of mathematical knowledge requires the use of the body as well, something that just recently, has been accepted, but not for all. We still see in the practice of education that many teachers and curricula designers conceive of mathematics as a purely intra-mental activity expressed in verbal form. This position has a long history. For instance, Plato wrote that "he who has got rid, as far as he can, of eyes and ears and, so to speak, of the whole body, these being... distracting elements which when they infect the soul hinder her from acquiring truth and knowledge" (Buchanan, 1976, p. 203). Mathematics, in Plato's epistemology is disembodied and would be the same even in the absence of human beings. Today, a famous follower of this way of thinking is the Fields Medal winner Alain Connes (for a fascinating discussion on this theme see Changeux and Connes, 1998).

In recent times though, as we have mentioned, the body has come to the fore in mathematics education. Research on gestures, for instance, shows this is the case. It is as if the brain were not enclosed in the head, but (not metaphorically) distributed across the body. An important insight that has taken root is that if we look deep into the meaning of a mathematical symbol—as in a process of *deconstruction*—we will find a bodily experience, an intuition. Of course this is not always an easy task because the structure of the reference field associated with a mathematical symbol is rather complex.

However, our work with digital media, especially with SimCalc, has shown that the mathematics of change and variation that in the past has been a black box for students, now can be approached in such a way that the mathematical structure behind change and motion tells a different story, a story in which the students finds mathematical understanding and identity. The classroom becomes a public scenario for discussion of ideas (closer to democracy than to the authoritarian Platonic epistemology) where students can compare their productions in an environment open to discussion with those of their classmates. A central feature of SimCalc is the potential to transform the socially disintegrated classroom into a participation space, where cognition is socially shared.

We have designed the simulation of a world to study change and variation, not through a classic analytic approach (where all is inert) but incorporating a dynamic narrative about motion and velocity. When the actor is in motion, we can simultaneously appreciate the corresponding Cartesian graph being born. It is a short cognitive distance for the student to imagine that, instead of the actor, it is herself who is walking and causing the corresponding Cartesian graph. Additionally, there is an emerging sense of mathematical identity in the design task as the student chose the analytic graph that should control the motion of the actor: The motion is controlled by the graph. *Motion is change*. This is the kernel of the grounding metaphor for the study of change. The mathematics of change and variation, of accumulation, is crystallized in the SimCalc universe.

The traditional discourse of the Calculus textbooks, supposedly delivers the opportunity to study the mathematics of change. However, instead of that, they offer a discourse whose tacit structure banishes change. This creates a rupture between the intuitively clear ideas of the mathematics of variation and change, as presented through *change is motion*, whose symbolic notation is controlled by basic motion metaphors (converging, oscillating, continuous, monotone behavior, etc.) and the formal structure whose *telos* is quite different: to create a *justification structure* based on the Arithmetization program of Weierstrass. It was Felix Klein, in 1896, who called this program the *Arithmetizing of Mathematics*. In this paper, Klein emphatically declares that, "it is not possible to treat mathematics exhaustively by the method of logical deduction alone, but that, even at the present time, intuition has its special province." A few years later, J. Pierpoint (1899) ended his paper on the arithmetization of mathematics with a feeling of melancholy clearly felt in his words:

> Built up on the simple notion of number, its truths are the most solidly established in the whole range of human knowledge. It is, however, not to be overlooked that the price paid for this clearness is appalling, it is *total separation from the world of our senses* (italics added, p. 406).

SimCalc learning environments provide students with a medium through which students can overcome the artificial difficulties that a premature arithmetization of the mathematics of change and variation can cast on the classroom. When the school does not hear the voice of the students, frequently uttered too low, the result has been to expel students from the Newtonian and Leibnizian paradise of Calculus. In response to this N. Luzin writes:

> What Weierstrass, Cantor, Dedekind did was very good. That is the way it *had* to be done. But whether this corresponds to what is in the depths of our consciousness is a very different question. I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work of the "justifications" and their "proofs" (Demidov and Shenitzer, 2000, p. 80).

In consequence, we have been witnessing a rupture, this time between a basic set of embodied conceptual mathematics, and its apparent formalization. The deep problem here is that Weierstrass Arithmetization ideas do not correspond to the formalization of the ideas of the mathematics of change and variation as embodied in SimCalc. In 1979, Jim Kaput explained that the meaning of mathematical operations was achieved through an essential projection from our internal cognitive experience onto the timeless, abstract-structural mathematical operations (Kaput and Clement, 1979). And more recently, Merlin Donald (2001) has provided a long-term perspective that cast light on the epistemological and didactical conflict we have been making explicit:

> *Humans thus bridge two worlds*. We are hybrids, half analogizers, with direct experience of the world, and half symbolizers, embedded in a cultural web. During our evolution we somehow supplemented the analogue capacities built into our brains over hundreds of millions of years with a symbolic loop through culture (p. 157).

Primary experiences are key for the students in their learning process. These experiences provide the roots of meanings.

It is clear that these reflections contribute to the perspective that has been called *embodied cognition*. According to Donald, it is our hybrid nature or rather, our *analogue half* that provides the (implicit) instructions for moving ourselves in the world of our experiences. This would not be possible if our knowledge of this vital space came from thought exclusively. Again, Donald (2001) provides the deep insight:

> Basic animal awareness intuits the mysteries of the world directly, allowing the universe to carve out its own image in the mind… In contrast, the symbolizing side of our mind… creates a sharply defined, abstract universe that is largely of its own invention (p. 155).

It is the human body moving in its (social) space that carries the seed for the process of symbolic abstractions. This is what we have tried to awaken in the SimCalc classroom mediated by forms of digital technology that cast light on cognition.

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